Maximizing coherence of oscillations by external locking

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(Dated: April 8, 2015)

We study how the coherence of noisy oscillations can be optimally enhanced by external locking. Basing on the condition of minimizing the phase diffusion constant, we find the optimal forcing explicitly in the limits of small and large noise, in dependence of phase sensitivity of the oscillator. We show that the form of the optimal force bifurcates with the noise intensity. In the limit of small noise, the results are compared with purely deterministic conditions of optimal locking.

PACS numbers: 05.40.Ca,05.45.Xt

Autonomous self-sustained oscillations may be extremely regular (like, e.g., lasers) or rather incoherent (like many biological oscillators, e.g., ones responsible for cardiac or circadian rhythms). A usual way to improve the quality of oscillations is to lock (synchronize) them by an external pacing [1, 2]. This is used in radio-controlled clocks and in cardiac pacemakers; also circadian rhythms are nearly perfectly locked by the 24-hours day/night force.

In this letter we address a question: which periodic force ensures, via locking, the maximal coherence of a noisy self-sustained oscillator? Of course, one has to fix the amplitude of the force, so the nontrivial problem is in finding the optimal force profile. We will treat this problem in the phase approximation [1], which is valid for general oscillators, provided the noise and the forcing are small. In this approximation the dynamics of the phase reduces to a noisy Adler equation [2, 7], and the maximal coherence is achieved if the diffusion constant of the phase is minimal. It should be noted that an optimal locking problem has been recently discussed for purely deterministic oscillations. There, the optimal condition was formulated as the maximal width of the Arnold's tongue (the synchronization region) or as the maximal stability of the locked state [3–6]. In our case there is an additional parameter, the noise intensity, and we will show that the optimal force profile depends on the noise amplitude. Below we will also compare the limit of small noise with purely deterministic setups.

Let us consider a self-sustained oscillator with frequency ω, its phase in presence of a small Gaussian white noise obeys the Langevin equation

$$\frac{d\varphi}{dt} = \omega + \beta^{-1/2}\xi(t), \qquad \langle \xi(t)\xi(t')\rangle = 2\delta(t-t'), \quad (1)$$

where β^{-1} is the noise intensity. A small periodic forcing with frequency Ω leads, in the first order in the force amplitude, to the following phase dynamics [1, 7]:

$$\frac{d\varphi}{dt} = \omega + s(\varphi)f(\Omega t) + \beta^{-1/2}\xi(t). \quad (2)$$

Here $s(\varphi)$ is the phase sensitivity function (a.k.a. phase response curve), and $f(\Omega t)$ is the phase-projected force term. Our goal will be to find such a forcing $f(\cdot)$ that maximizes the coherence, i.e. minimizes the diffusion constant of the phase φ. This optimal force will depend on the phase sensitivity function $s(\cdot)$ and on the noise intensity β.

As the first step we introduce the slow phase $\phi = \varphi - \Omega t$ and perform the standard averaging over the period $2\pi\Omega^{-1}$ [1, 2], this yields

$$\frac{d\phi}{dt} = \omega - \Omega + g(\phi) + \beta^{-1/2}\xi(t) = -\frac{dv(\phi)}{d\phi} + \beta^{-1/2}\xi(t), \quad (3)$$

where

$$g(\phi) = \frac{1}{2\pi}\int_0^{2\pi} dy s(\phi + y)f(y), \quad (4)$$

and we introduced the "potential"

$$v(\phi) = (\Omega - \omega)\phi - \int^{\phi} g(y)\, dy. \quad (5)$$

Let us consider a situation, where the mean frequency of oscillations is exactly that of the forcing; this means that the slow phase ϕ performs a random walk without a bias. This happens for a purely periodic, non-inclined potential. This condition, as it follows from (5), defines the optimal frequency of the forcing

$$\overline{\Omega} = \omega + \langle s\rangle\langle f\rangle, \quad (6)$$

where we denote $\langle f(\phi)\rangle = (2\pi)^{-1}\int_0^{2\pi} f(\phi)\, d\phi$. Thus, without loss of generality we can assume that $\langle s\rangle = \langle f\rangle = 0$ and $\Omega = \omega$.

The problem of finding the diffusion constant D of a particle in a periodic potential v, driven by a white Gaussian noise, has been solved in Ref. [8] (and generalized to the case of an inclined potential in Ref. [9]):

$$D = \frac{D_0}{\langle \exp(\beta v)\rangle\langle \exp(-\beta v)\rangle}, \quad (7)$$

where D_0 is the bare diffusion constant without potential. Thus, the problem of maximizing the coherence reduces to maximizing the expression

$$C = \langle \exp(\beta v)\rangle\langle \exp(-\beta v)\rangle. \quad (8)$$

As an additional condition we have to fix the intensity of the force:

$$\langle f^2 \rangle = const. \tag{9}$$

The formulated optimization problem is quite complex to be solved in general. Therefore, below we consider some simplifying cases, and will perform a rather full analysis for a simple bi-harmonic phase sensitivity function. The main feature we will focus on, are bifurcations in dependence on the form of this function and on the noise intensity; we will see that different forcing waveforms provide optimal coherence in different domains of the parameter space.

For the analytical consideration below it is convenient to use Fourier transforms, which we will denote by capitals:

$$s(x) = \sum_k S_k \exp[ikx], \quad S_k = \frac{1}{2\pi}\int_0^{2\pi} s(x)\exp[-ikx]dx, \tag{10}$$

and the same for functions f, g, v, Fourier harmonics of which we denote as F_k, G_k, V_k, respectively. Because $g(\phi)$ is according to (4) a convolution of f and s, and v is the integral of g, we have

$$G_k = S_k F_{-k}, \qquad V_k = ik^{-1} S_k F_{-k}. \tag{11}$$

The condition on the norm of the force (9) now reads

$$\sum_k |F_k|^2 = const. \tag{12}$$

We start with the case of strong noise (small β). Expanding (8), we obtain a simple expression for the quantity to be maximized:

$$C \approx 1 + \beta^2\langle v^2\rangle = 1 + \beta^2 \sum_k k^{-2}|S_k|^2|F_k|^2. \tag{13}$$

Together with condition (12), the maximum can be found by virtue of Lagrange multipliers:

$$|F_k| \sim \delta_{k,K}, \quad \text{where} \quad K = \arg\max(k^{-2}|S_k|^2). \tag{14}$$

Thus, for large noise, the optimal forcing is purely harmonic one $f(x) \sim \cos(Kx)$, where K is determined from (14).

The case of small noise is the limit $\beta \to \infty$. In this case the integrals in the expression (8) can be asymptotically estimated as Laplace integrals:

$$\begin{aligned}\langle \exp(\beta v)\rangle &\approx (2\pi)^{-1}\exp(\beta v_{max}),\\ \langle \exp(-\beta v)\rangle &\approx (2\pi)^{-1}\exp(-\beta v_{min}),\end{aligned} \tag{15}$$

what gives

$$C \sim \exp(\beta(v_{max} - v_{min})). \tag{16}$$

Suppose now that $v_{min} = v(x_2)$ and $v_{max} = v(x_1)$. Then

$$\begin{aligned}\ln C \sim v_{max} - v_{min} &= \int_{x_1}^{x_2} g(\phi)\, d\phi = \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \int_{x+x_1}^{x+x_2} s(y)dy.\end{aligned} \tag{17}$$

Using additionally condition (9) with a Lagrange multiplier, we obtain

$$f(x) = \text{const}\int_{x+x_1}^{x+x_2} s(y)dy. \tag{18}$$

Substituting this into conditions $g(x_1) = g(x_2) = 0$, we get an equation for $\Delta = x_2 - x_1$ (only this difference is important, but not the values of x_1, x_2):

$$\begin{aligned}p(\Delta) &= \int_0^{2\pi} s(z)\int_0^{\Delta} s(z+y)\, dz\, dy = \\ &= 2\pi\sum_k k^{-1}|S_k|^2 \sin k\Delta = 0.\end{aligned} \tag{19}$$

This equation has always a solution $\Delta = \pi$, but depending on the form of the phase sensitivity function s there can be other solutions, corresponding to local maxima of C; one has to compare different possible values of Δ to find the global maximum. Once Δ is found, the corresponding force can be expressed as

$$f(\phi) \sim \int_{\phi}^{\phi+\Delta} s(y)\, dy, \qquad F_k \sim S_k\frac{\exp[ik\Delta]-1}{ik}. \tag{20}$$

Here below we present the simplest nontrivial example, where it is possible, in addition to the asymptotic cases of small and large noise considered above, to perform the analysis for intermediate noise levels. We consider the bi-harmonic phase sensitivity function

$$s(x) = 2\sqrt{q}\cos x + 2\sqrt{1-q}\cos 2x, \tag{21}$$

where parameter q describes the relative weight of the harmonics.

The limit of strong noise (14), with $|S_1|^2 = q$, $|S_2|^2 = 1-q$, yields

$$f(x) \sim \begin{cases}\cos 2x & \text{if } 0 \le q < 1/5,\\ \cos x & \text{if } 1/5 < q \le 1.\end{cases} \tag{22}$$

The limit of weak noise leads to the following expression for function (19):

$$p(\Delta) = q\sin\Delta + \frac{1-q}{2}\sin 2\Delta. \tag{23}$$

For $q > 1/2$, the only root in (23) is $\Delta = \pi$, while for $q < 1/2$ there is an additional root $\Delta_1 = \arccos(-q/(1-q))$. Substituting this into (20), we obtain for small noise

$$|F_1|^2 = 1 - |F_2|^2 = \begin{cases}2q & \text{if } 0 \le q < 1/2,\\ 1 & \text{if } 1/2 < q \le 1.\end{cases} \tag{24}$$

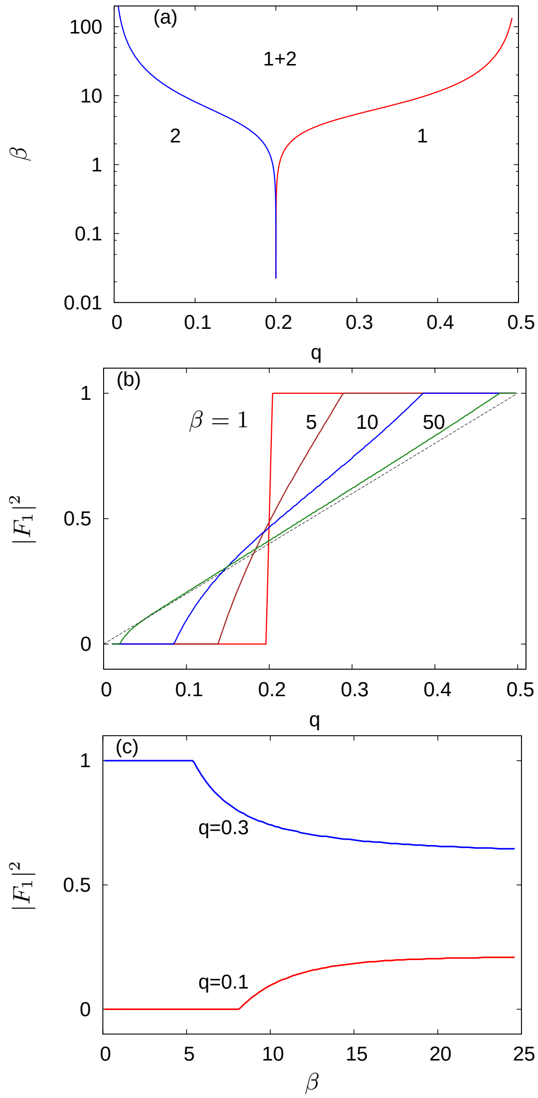

FIG. 1. (color online) (a) Domains on the plane of parameters (q, β) where the optimal force has one harmonics (2: the second one, 1: the first one), and two harmonics (1+2), according top expressions (27,28). (b) The intensity of the first harmonics $|F_1|^2$ as a function of q for different noise intensities β. Thin dashed black line shows the limit of small noise (24). (c) Two dependencies of $|F_1|^2$ on the noise intensity β showing bifurcations from one-mode to two-mode solutions at critical values of β.

Let us now consider general noise intensities. The forcing in this case should be also generally bi-harmonic (higher harmonics disappear according to (11)):

$$f(x) = a\cos x + b\cos 2x + c\sin 2x, \tag{25}$$

with unknown constants a, b, c satisfying $a^2+b^2+c^2 = 1$. In this representation the potential $v(x)$ reads

$$v(x) = -\sqrt{q}a\sin x - \frac{\sqrt{1-q}b}{2}\sin 2x - \frac{\sqrt{1-q}b}{2}\cos 2x. \tag{26}$$

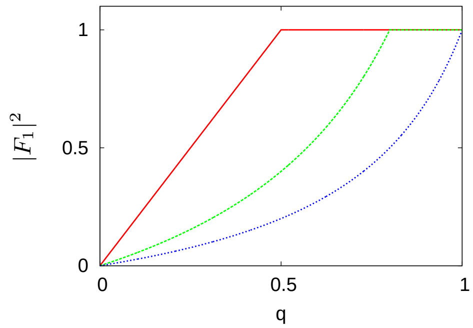

FIG. 2. (color online) The intensity of the first harmonic component in the optimal force as function of parameter q, for three optimization criteria: top solid red line: maximal coherence in the weak noise limit (24); middle dashed green line: maximal width of the synchronization region (30); bottom dotted blue curve: maximal linear stability of the locked state (29).

Unfortunately, after substitution of this potential in the expression (8) for the factor C, we obtain integrals which cannot be expressed in a closed analytic form. However, for a purely first-harmonic forcing ($b = c = 0$) and a purely second-harmonic forcing ($a = 0$), the factor C as well as its derivatives can be expressed via first order Bessel functions. Thus, it is possible to find the domains of stability of these pure forcing terms analytically, for arbitrary values of noise intensity β. These lengthy but straightforward calculations give the stability boundaries in a parametric form: The first-harmonic force loses stability at the curve on the (β, q) plane given according to

$$q = \frac{z(-I_4(z)+I_0(z))}{8I_1(z)+z(-I_4(z)+I_0(z))}\ , \quad \beta = \frac{z}{\sqrt{q}}. \tag{27}$$

The stability boundary of the second-harmonic solution is

$$q = \frac{I_1(z)}{I_1(z)+2zI_0(z)}\ , \quad \beta = \frac{2z}{\sqrt{1-q}}. \tag{28}$$

We illustrate these domains in Fig. 1. Here we also show numerically obtained dependencies of $|F_1|^2$ (the intensity of the second harmonis is $|F_2|^2 = 1-|F_1|^2$) on parameters q and β, demonstrating bifurcations on the form of the forcing.

Next we discuss a relation between different criteria used for the "optimal locking". While here we optimize the coherence in the presence of noise, in Refs. [3, 4] purely deterministic criteria have been suggested. It is instructive to compare them with our approach in the limit of small noise. Suppose that the coupling function $g(\phi)$ has zeros at $\phi_{1,2}$ (where ϕ_1 is the stable one), and extrema at $\phi_{3,4}$. In the approach of [4], the linear stability at the stable equilibrium $|g'(x_1)|$ is maximized. In the approach of [3], the width of the synchronization region $\sim |g(\phi_3) - g(\phi_4)|$ is maximized. In our maximization of the coherence, the potential barrier for a noise-induced

phase slip $\sim \left|\int_{\phi_1}^{\phi_2} g(x)dx\right|$ should be maximal. For the discussed above example of a bi-harmonic phase sensitivity function (21), all the optimal forcings can be found analytically, they are generally also bi-harmonic. The approach of [4] yields in this case

$$|F_1|^2 = 1 - |F_2|^2 = q/(4-3q), \tag{29}$$

while the approach of [3] gives

$$|F_1|^2 = 1 - |F_2|^2 = \begin{cases} \frac{2q}{4-3q} & \text{if } 0 \le q < 4/5, \\ 1 & \text{if } 4/5 < q \le 1. \end{cases} \tag{30}$$

We compare the results in Fig. 2. One can see that for the minimal coherence, presence of a strong first harmonics component in the forcing is more important than for other criteria.

In conclusion, we have studied the problem of maximizing coherence of oscillations by external locking, in the phase approximation. The optimal phase forcing function depends not only on the phase sensitivity function of the system, but also on the noise intensity. For large noise a purely harmonic forcing is optimal, the number of the harmonic depends on the phase sensitivity. For smaller noise, a bifurcation to a more general, multi-harmonic forcing may occur. We have also demonstrated, that different optimality conditions in the purely deterministic case lead to different optimal forcing functions, which also differ from the limit of small noise when optimization of the coherence is performed.

The author thanks the Galileo Galilei Institute for Theoretical Physics, Florence, Italy, for the hospitality and the INFN for partial support during the completion of this work.

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